NOV 11 1936
D-1

# NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XI BATON ROUGE, LA., NOVEMBER, 1936 No. 2

*Our Problem Department*

*Mathematics and the Hypotheses of Science*

*Expansion of a Function in Terms of Exponentials*

*Humanism and History of Mathematics*

*The Teacher's Department*

*Mathematical Notes*

*Book Reviews*

*Advertisements*

PUBLISHED BY LOUISIANA STATE UNIVERSITY

Every paper on technical mathematics offered for publication should be submitted with return postage to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

1. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the Magazine.

Papers intended for the Teacher's Department, Department of History of Mathematics, Book Reviews or Problem Department should be sent to the respective Chairmen.

*Committee on Algebra and Number Theory*: L. E. Bush, W. Vann Parker, James McGiffert.

*Committee on Analysis and Geometry*: W. E. Byrne, Wilson L. Miser, Dorothy McCoy, H. L. Smith.

*Committee on Teaching of Mathematics*: Joseph Seidlin.

*Committee on Statistics*: C. D. Smith, Irby C. Nichols.

*Committee on Mathematical Notes*: L. J. Adams.

*Committee on Book Reviews*: P. K. Smith.

*Committee on Problem Department*: T. A. Bickerstaff.

*Committee on Humanism and History of Mathematics*—G. Waldo Dunnington.

*Published by the Louisiana State University Press*

*Club Rates May Be Had on Application*

*Subscription, $1.50 Per Year in Advance Single Copies, 20c.*

Vol. XI BATON ROUGE, LA., NOVEMBER, 1936 No. 2

Published 8 Times Each Year by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

All Business Communications should be addressed to the Editor and Manager P. O. Box 1322, Baton Rouge, La.

EDITORIAL BOARD

S. T. SANDERS, Editor and Manager, P. O. Box 1322, Baton Rouge, La

T. A. BICKERSTAFF
University of Mississippi
University, Mississippi

W. VANN PARKER,
Louisiana State University
Baton Rouge, La.

G. WALDO DUNNINGTON
University of Illinois
Urbana, Illinois

JOS. SEIDLIN
Alfred University
Alfred, New York

L. E. BUSH
College of St. Thomas
St. Paul, Minn.

H. LYLE SMITH
Louisiana State University
Baton Rouge, La.

WILSON L. MISER
Vanderbilt University
Nashville, Tennessee

IRBY C. NICHOLS
Louisiana State University
Baton Rouge, La.

JAMES McGIFFERT
Rensselaer Poly. Institute
Troy, New York

W. E. BYRNE
Virginia Military Institute
Lexington, Virginia

C. D. SMITH
Mississippi State College
State College, Miss.

DOROTHY McCOY
Belhaven College
Jackson, Miss.

L. J. ADAMS
Santa Monica Junior College
Santa Monica, Cal.

P. K. SMITH
Louisiana Polytechnic Institute
Ruston, Louisiana

*This Journal is dedicated to the following aims:*

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

# Our Problem Department

The brilliant chairman of our Problem Department, T. A. Bickerstaff, writes that owing to his recent assumption of the duties of Registrar along with those of teacher of mathematics at the University of Mississippi, he will no longer be able to function effectively as head of this Department. We have received with much regret this announcement. Our profound appreciation is due him for his past years of service to the cause of the News Letter and the Magazine. While the burden is thus imposed on us of naming a new chairman for this Department, we shall with Professor Bickerstaff's permission retain him as advisory member of the Committee on problem material.

Preparatory to what appears to be an enforced reorganization of the Department, we are omitting it altogether from the present issue.

With no desire to conceal from our readers any plans or policies editorially approved for the expansion of the Magazine, we here frankly declare that we desire to fill the vacated chairmanship with one who along with a competent scholarship and adequate mathematical ability has also a high vision of the possibilities of a finely maintained Problem Department.

Consistent with a policy heretofore observed, to develop many widely separated centres of administrative responsibility for the Magazine, is our particular ambition that the new Problem Department chairman should come from territory of the United States not presently represented on the Editorial Board.

Until election of the new chairman shall be made, we request that all proposed problems, or solutions of problems already proposed in the Magazine, be sent to W. Vann Parker, Louisiana State University, Baton Rouge, La.

S. T. SANDERS.

# Mathematics and the Hypotheses of Science*

By H. J. ETTLINGER
*The University of Texas*

The role of numbers and quantitative thinking in all the various branches of the natural sciences is admittedly very fundamental. The very interesting interplay and interrelation between the basic ideas of mathematics and our attempts to explain, interpret and predict the phenomena of nature, form a profoundly romantic chapter of the development of man's knowledge of the world in which he lives. The old biological saw: which came first, the hen or the egg, has its counterpart in this situation with some variations. The question as to which can lay claim to the priority of credit, the mechanisms, i. e., the structure of mathematical thought or the hypothesis, i. e., the rock bottom substrata of the sciences, must be answered in a two-fold manner. In the first place, numerous examples may be cited where the needs of science have led to striking advances in mathematics. On the other hand, one may point out the more recent profound influence of the concepts of pure mathematics on the whole outlook of the physical sciences. The situation parallels closely a like state of affairs within each field of science with respect to the marking off of the province of the pure from the realm of the applied. Nay, in some instances, with respect to the relation between mathematics and the other sciences it is difficult to determine the exact line of demarcation, and, in fact, this division may be said to be non-existent. That is to say, there are border line cases encountered in such a survey which almost force one to take refuge in the statement of Sir James Jeans, British astronomer and physicist, when he asserted that all aspects of knowledge with a capital K are applications of mathematics and preached of God the Mathematician. Without deifying or even capitalizing mathematics, our examination of the interrelations which it bears with the other streams of scientific thought will I trust be very fruitful.

The early history of the counting number system is undoubtedly intimately connected with the needs for a census of the tribe and its

*Address of the retiring President of the University of Texas Chapter of the Society of Sigma Xi, delivered before the annual banquet and initiation, on May 12, 1936.

possessions. Likewise, Euclid's systematic presentation of his Elements was preceded by long usage of practical rules of mensuration. In this way the elementary needs of social knowledge called into being the beginnings of number ideas and geometry. Almost as old as the hypothesis that all material bodies were composed of the four constituents, air, earth, fire and water is the one which puts forth the thesis that nature "abhors a vacuum." A classic experiment of elementary physics is that of the Magdeburg sphere in which two hemispheres are placed so as to form a complete sphere. If the air within is partially exhausted, "nature's effort to fill this vacuum results in the pressure on the surface on the hemispheres." All suction effects were explained from this point of view.

The endeavor to make Nature completely continuous resulted in a number of interesting developments. The great Newton had introduced among other things two discontinuous ideas—the corpuscular theory of light and action at a distance. According to the first hypothesis, light was transmitted by a stream of particles proceeding from the source of the light and activating the retina. By the second hypothesis the motion of the bodies of the solar system as defined by Kepler's laws was explained. This latter hypothesis assumed that every particle in the universe attracted every other particle by a force which was inversely proportional to the square of the distance between them. This description clearly violated the laws of libel with regard to the respectability of nature and seemd to introduce magic and miracle into science. To reconcile these inconsistencies, the light-bearing ether was invented, an all-pervading, perfectly elastic and genuinely ideal substance, which filled all of empty space, not only that between the heavenly bodies, but also that between their junior earthly offspring, the molecules and atoms. Thus was connective tissue supplied for the gaunt skeleton of discrete corpuscles and disjointed attracting centers.

The undulatory or wave theory of light whose chief forebear was Huygens and the electromagnetic field theory of Maxwell were descendants in a direct line of the efforts to wipe out the apparent emptiness of the earlier basic hypotheses. The brilliant Faraday had experimentally detected lines of electric and magnetic force extending continuously from charged body to charged body. Maxwell, using this physical hypothesis of continuous connections, set up a mathematical description of space which assumed the presence of an electric and a magnetic force at each point. These forces were interpreted as due to stresses and strains in the ether. When we realize how closely identified the ether was in all its properties with empty space, we do not marvel

that out of the mathematics of Maxwell came the prophecy of the detection of electric waves later produced and found by Hertz. To summarize the distinction between the Faraday-Maxwell continuous medium point of view and that of the earlier workers, we quote from the Preface to the First Edition of Maxwell's classic treatise of Electromagnetism. Maxwell writes, "For instance, Faraday, in his mind's eye saw lines of force traversing all space where the mathematicians saw centers of force attracting at a distance: Faraday saw a medium, where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids."

In Maxwell's times, though continuity was definitely in fashion, sharp working definitions of this property were not in general use. However, it remained for Oliver Heaviside, without the background of any orthodox mathematical training to be the first natural scientist to make effective and wide spread use of discontinuity in connection with a mathematical situation. Heaviside, living as a recluse and preaching a strange doctrine that Truth dwelt in a bottomless pit, was concerned with the transfer of voltage and current in cables. He met with astonishing success in the use of a number which was discontinuous and which had the following life history: its past was completely described by the value zero, its present and future by the value one. He called it, in fact, the unit function. Exactly how he came to use this mathematical artifice or ingenious device is difficult to determine. The combination of the unit function with a daring symbol for operating with this number to obtain other useful numbers was not only unappreciated by his contemporaries but met with actual hostility. He was a Bolshevist in the eyes of orthodox scientists, his writings were completely neglected, and his declining years were darkened by the disfavor of the great Lord Kelvin. That Heaviside's ideas were fruitful is attested by the fact that the late Professor Michael Pupin received one million dollars from the A. T. and T. Co. for a device arising from one of Heaviside's ideas which Pupin rediscovered independently twenty-five years later.

From a slightly different angle the contrast in points of view of the hypotheses of continuous versus discontinuous is exemplified in the problem of the transverse, to and fro motions, of a stretched string. Here a solution was obtained mathematically by Euler and the Bernoullis on the older hypothesis of a string as a continuous distribution of matter. It was Lord Rayleigh who introduced into the picture the aspect of molecular structure which emphasized the dis-

continuous aspect of matter as a large collection of distinct elements or beads. Rayleigh's principle was another example of the bringing together of two apparently different hypotheses of continuity and discontinuity and fusing them into one. This aspect is a very fundamental one today and both the continuous string and the string of beads have proved useful. For a sufficiently large number of beads the results coalesce and from the purely practical or computational side the continuous model has a distinct advantage over the large number.

Essentially the same considerations enter into the more familiar situation of the concept of the density of a material body. The usual definition is as follows: consider a sequence of spheres with a fixed center whose radii steadily decrease to the limit zero. Divide the mass of each sphere by its volume. This is called the average density over the sphere. The limit of the succession of the average densities is called the density at the point, provided there is such a limit. We know, however, that a "common garden variety" piece of iron will not give a result for the above process. Up to a certain stage in this process of taking smaller and smaller samples we obtain a result usually called the density of iron but when we proceed beyond molecular dimensions the so-called density number may fall to zero for points between molecules or for points inside the atom. Inside of an electron or inside a proton the average density may jump to a value many thousand fold larger than what is ordinarily called the density of iron. Nevertheless, in a great many cases we find it convenient and useful to replace the atomistic picture by such a continuous, homogeneous and uniform spread. This is distinctly a case of replacement of a given definite situation by an average one and illustrate how the statistical point of view is of such profound influence in so-called exact situations.

More recently, the atomics of material distributions of physics and chemistry have been carried over into the field of mechanics. About 1900 Planck announced the quantum hypothesis of energy. The fundamental tenet of this hypothesis is that there is a smallest amount of energy in nature which is a basic unit of energy corresponding to the atom in material bodies. This hypothesis which introduced the discontinuous or discrete into mechanics was followed by a very striking reconstruction of the entire field of small scale phenomena. Even motion itself and time have come under suspicion and there are some who are wondering whether Zeno's paradox may not find its way into the quantum mechanics of the near future. For example, Zeno argued that motion was impossible since at each instant of time the point supposed to be in motion must be in a definite position and hence

it could not move. When the notion of continuity was introduced, this paradox was resolved in the Newtonian calculus setting of fluxions. However, it is quite possible that the quantizing of space and time, that is, the hypothesis of a smallest unit of time and of space, would permit of a resolution and interpretation of Zeno's paradox in an entirely different manner. In the early nickelodean days before movies, there were penny-in-the-slot machines with devices for showing with considerable rapidity a succession of views either scenic or dramatic. These in effect were the ancestors of the present day cinematograph. Clearly, in this case, motion consists of a sequence of positions fixed in space for short intervals of time different from the continuous change of position. Successive positions are determined by adding definite units of time and space, and a diagram of such a motion would register pieces of lines which may not be connected.

The possible quantizing of space and time recalls to mind that perhaps the most striking example of the influence of a mathematical set-up on physical hypotheses is the space-time frame of reference proposed by Einstein into which natural phenomena are to be placed. We may recall that preceding Einstein, the pure mathematicians, Bolyai, Lobachevsky, and Riemann had developed systems of geometry which departed radically in their initial assumptions from that of Euclid, and hence were called non-Euclidean geometries, in which the important elements were not necessarily straight lines but might be curves. Also the arithmetic of numbers with many constituent parts, what is called tensor analysis, had been worked out and all of this without much hope of immediate application. From this mathematical setting, the following important conclusions amongst others were drawn by Einstein:

a) that mass and energy are expressible in terms of each other,

b) that mass, length, and time units depend on the observer,

c) that while empty space is Euclidean the space in the neighhood of mass is modified into a warped or curved space,

d) that many of the physical and mechanical properties due to the presence of mass can be embodied in the numbers describing the space, i. e., in the local geometry. In addition Einstein and others also made use of the concept that a group of operations of specified type will in general leave certain numbers unchanged, the so-called invariants of the group. It may be readily seen how closely this is connected with the hypothesis that the laws of nature are and must be independent of the particular units or numbers used by a given observer.

In setting up a geometry or a number system, or both, in mathematics, for the purpose of characterizing the intrinsic properties of a system, it is customary to start with a collection of undefined elements and to prescribe a set of assumptions or postulates to be satisfied by these elements. This axiomatic branch of the very foundations of mathematics regarded by uninitiated as most abstract has its uses in the field of applied science. We need but recall how the formulation of our results concerning the flow of heat has substantially used this concept of the phenomenon of heat as an undefined element, and we cite the fact that these results have held forth equally well when heat was regarded as a perfect fluid, caloric, down to our modern mechanical interpretation of heat as the dance of the molecules. In fact, our scientific body of results on conduction of heat, flow of perfect fluids, diffusion in gases and liquids, and propagation of electric waves achieves a remarkable unity when we treat them at once by the use of undefined elements. This serves to bring out more clearly the remarkable contribution of Fourier who in his Theory of Heat introduced the analysis of a general set of numbers into harmonics or waves. From the invention of the calculus by Newton and Leibnitz, to the time of Einstein no one idea had a more profound influence on subsequent developments in mathematics as well as in the natural sciences than this work of Fourier. He made use of a set of numbers connected with wave diagrams which possess the property of regularity and periodicity. Fourier showed that even though these elements were continuous themselves, they could be used to represent discontinuous numbers. Into many branches of knowledge these ideas have penetrated.

In spite of frequent betrayal by our sense of sight, workers in science are habituated to leaning heavily on a mechanical or physical model. May it not be possible that we have nearly reached the frontier in our efforts to picture the inner structure of the organic and inorganic world and that we must come to rely more and more on the not too closely circumscribed potentialities of present day mathematics or that which is still to come. Clearly physical and mechanical models have served nobly but it may be that the inner complexities of the atom and cell may call for a more comprehensive mathematical model than has been offered heretofore.

Just one more of the many interrelations will now be presented, in which we have a special interest. Today the foundation stones of logic, essential to all exact thinking, are being critically examined. Logic, together with the concept of causality form the very rock bottom upon which scientific results rest. Electronic situations have introduced what is called the Heisenberg Law of Indeterminacy which has

reference to observations made on moving electrons. This law assumes that the very act of observation introduces uncertainty in the quantities observed so that one may determine either the position or the momentum of an electron with precision but not both. To some, principally non-scientific, persons this state of affairs reduces all of scientific thought to chaos and transforms conclusions to haphazard guesses. It is here that a mathematical model may be extremely useful. We may set up the problem as follows: We specify the initial state of a point (particle) exactly as to position and momentum and prescribe an exact causal law as represented by an equation. We find that an infinity of solutions is possible. But, we have found this result, that in the neighborhood of every case of indeterminacy, there exist exact solutions which will serve to approximate the given case of indeterminacy. Even as we may on the average replace the discontinuous by the continuous, under suitable restrictions, so may we in similar fashion regard the indeterminate as approximate to the determinate.

This brings us back to the title. We do not pretend to have covered completely the important interrelationship of mathematics and the hypotheses of science. Close cooperation between workers in scientific projects is very much in the air today. It is in this spirit that this partial picture has been presented of how mathematics has influenced and been influenced by the other sciences.

# Expansion of a Function in Terms of Exponentials

By LYNN L. MERRILL
*Rensselaer Polytechnic Institute*
*Troy, New York*

Let it be required to develop a function of $x$ in successive integrals of $e^x$.

Assume

$$f(x)=a_0e^x+a_1\int_0^x e^x dx+a_2\int_0^x\int_0^x e^x dxdx+\dots$$

$$+a_n\underbrace{\int_0^x\dots\int_0^x}_{\leftarrow n\rightarrow} e^x dx\dots dx+\dots,$$

then

$$f'(x)=a_0e^x+a_1e^x+a_2\int_0^x e^x dx+a_3\int_0^x\int_0^x e^x dxdx+\dots.$$

$$f''(x)=a_0e^x+a_1e^x+a_2e^x+a_3\int_0^x e^x dx+a_4\int_0^x\int_0^x e^x dxdx+\dots.$$

$$f'''(x)=a_0e^x+a_1e^x+a_2e^x+a_3e^x+a_4\int_0^x e^x dx+\dots.$$

etc.

Set $x=0$. Then

$$f(0)=a_0 \text{ or } a_0=f(0)$$

$$f'(0)=a_0+a_1 \text{ or } a_1=f'(0)-f(0)$$

$$f''(0)=a_0+a_1+a_2 \text{ or } a_2=f''(0)-f'(0)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f^{(n)}(0)=a_0+a_1+\dots+a_n \text{ or } a_n=f^{(n)}(0)-f^{(n-1)}(0)$$

$$\therefore f(x)=f(0)e^x+[f'(0)-f(0)]\int_0^x e_x dx+[f''(0)-f'(0)]\int_0^x\int_0^x e^x dxdx+\dots$$

$$(1)\qquad +[f^{(n)}(0)-f^{(n-1)}(0)]\underbrace{\int_0^x\dots\int_0^x}_{\leftarrow n\rightarrow} e^x dx\dots dx+\dots.$$

This series may be reduced to MacLaurin's series by observing that

$$\int_0^x e^x dx = e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

$$\int_0^x \int_0^x e^x dx dx = e^x - x - 1 = \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

$$\int_0^x \int_0^x \int_0^x e^x dx dx dx = e^x - \frac{x^2}{2!} - x - 1 = \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

etc.

Then $f(x)$ becomes

$$f(x) = f(0)\left(1 + x + \frac{x^2}{2!} + \dots\right) + [f'(0) - f(0)]\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$$

$$+ [f''(0) - f'(0)]\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \dots.$$

$$= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots.$$

If it be assumed that

$$f(x) = a_0 e^{x-a} + a_1 \int_0^{x-a} e^x d + x a_2 \int_0^{x-a} \int_0^x e^x dx dx$$

$$+ a_3 \int_0^{x-a} \int_0^x \int_0^x e^x dx dx dx + \dots + a_n \underbrace{\int_0^{x-a} \int_0^x \dots \int_0^x}_{\leftarrow n \rightarrow} e^x dx \dots dx + \dots$$

$$f'(x) = a_0 e^{x-a} + a_1 e^{x-a} + a_2 \int_0^{x-a} e^x dx + a_3 \int_0^{x-a} \int_0^x e^x dx dx + \dots.$$

etc.

Then

$$a_0 = f(a)$$

$$a_1 = f'(a) - f(a)$$

$$a_2 = f''(a) - f'(a)$$

$$\dots\dots\dots\dots\dots\dots$$

$$a_n = f^{(n)}(a) - f^{(n-1)}(a),$$

and

$$f(x)=f(a)e^{x-a}+[f'(a)-f(a)]\int_0^{x-a}e^x dx$$
$$+[f''(a)-f'(a)]\int_0^{-a}\int_0^{x}e^x dxdx+\dots.$$
$$(2)\qquad +[f^{(n)}(a)-f^{(n-1)}(a)]\underbrace{\int_0^{x-a}\int_0^{x}\dots\int_0^{x}}_{\leftarrow n\rightarrow}e^x dx\dots.dx+\dots.$$

If
$$x=a+h$$
$$x-a=h$$

and $f(x)$ becomes

$$f(a+h)=f(a)e^h+[f'(a)-f(a)]\int_0^{h}e^x dx$$
$$+[f''(a)-f'(a)]\int_0^{h}\int_0^{x}e^x dxdx+\dots.$$
$$(3)\qquad +[f^{(n)}(a)-f^{(n-1)}(a)]\underbrace{\int_0^{h}\int_0^{x}\dots\int_0^{x}}_{\leftarrow n\rightarrow}e^x dx\dots.dx+\dots.$$

To find the remainder, take

$$f(a+h)=f(a)e^h+[f'(a)-f(a)]\int_0^{h}e^x dx+\dots.$$
$$+[f^{(n)}(a)-f^{(n-1)}(a)]\underbrace{\int_0^{h}\int_0^{x}\dots\int_0^{x}}_{\leftarrow n\rightarrow}e^x dx\dots.dx$$
$$+P\underbrace{\int_0^{h}\int_0^{x}\dots\int_0^{x}}_{\leftarrow (n+1)\rightarrow}e^x dx\dots.dx.$$

where $P$ is to be determined.

Assume

$$\phi(x)=f(A)-f(x)e^{A-x}-[f'(x)-f(x)]\int_0^{A-x}e^x dx$$
$$-[f''(x)-f'(x)]\int_0^{A-x}\int_0^{x}e^x dxdx\dots$$

$$-[f^{(n)}(x)-f^{(n-1)}(x)]\underbrace{\int_0^{A-x}\int_0^{x}\cdots\int_0^{x}}_{\longleftarrow n\longrightarrow}e^x dx\ldots.dx$$

$$-P\underbrace{\int_0^{A-x}\int_0^{x}\cdots\int_0^{x}}_{\longleftarrow (n+1)\longrightarrow}e^x dx\ldots.dx.$$

where $A=a+h$

or $A-a=h.$

When $x=A, \quad \phi(x)=0.$

When $x=a, \quad \phi(x)=0.$

Therefore

$$\phi'(x)=0 \text{ for some value X, } a<X<A.$$

$$\phi'(x)=f(x)e^{A-x}-f'(x)e^{A-x}+[f'(x)-f(x)]e^{A-x}-[f''(x)-f'(x)]\int_0^{A-x}e^x dx$$

$$+[f''(x)-f'(x)]\int_0^{A-x}e^x dx-[f'''(x)-f''(x)]\int_0^{A-x}\int_0^{x}e^x dx dx+\ldots.$$

$$+[f^{(n)}x-f^{(n-1)}(x)]\underbrace{\int_0^{A-x}\int_0^{x}\cdots\int_0^{x}}_{\longleftarrow n-1\longrightarrow}e^x dx\ldots.dx$$

$$-[f^{(n+1)}(x)-f^{(n)}(x)]\underbrace{\int_0^{A-x}\int_0^{x}\cdots\int_0^{x}}_{\longleftarrow n\longrightarrow}e^x dx\ldots.dx$$

$$+P\underbrace{\int_0^{A-x}\int_0^{x}\cdots\int_0^{x}}_{\longleftarrow n\longrightarrow}e^x dx\ldots.dx.$$

$\therefore$ for $\phi'(x)$ to be zero

$$P=[f^{(n+1)}(X)-f^{(n)}(X)].$$

Let $X=a+\Theta h,\ 0<\Theta<1.$

Then remainder is

$$R_1=[f^{(n+1)}(a+\Theta h)-f^{(n)}(a+\Theta h)]\underbrace{\int_0^{h}\int_0^{x}\cdots\int_0^{x}}_{\longleftarrow n+1\longrightarrow}e^x dx\ldots.dx.$$

If it be assumed that

$$f(x)=a_0e^{-x}+a_1\int_0^x e^{-x}dx+a_2\int_0^x\int_0^x e^{-x}dxdx+\ldots.$$
$$+a^n\underbrace{\int_0^x\ldots.\int_0^x}_{\longleftarrow n\longrightarrow} e^xdx\ldots.dx+\ldots.,$$

Then by the same method as used above,

$$f(x)=f(0)e^{-x}+[f'(0)+f(0)]\int_0^x e^{-x}dx$$
$$+[f''(0)+f'(0)]\int_0^x\int_0^x e^{-x}dxdx+\ldots.$$
$$(4)\qquad +[f^{(n)}(0)+f^{(n-1)}(0)]\underbrace{\int_0^x\ldots.\int_0^x}_{\longleftarrow n\longrightarrow} e^{-x}dx\ldots.dx+\ldots.$$

and

$$f(x)=f(a)e^{-(x-a)}+[f'(a)+f(a)]\int_0^{x-a} e^{-x}dx$$
$$+[f''(a)+f'(a)]\int_0^{x-a}\int_0^x e^{-x}dxdx+\ldots.$$
$$(5)\qquad +[f^{(n)}(a)+^{(n-1)}f(a)]\underbrace{\int_0^{x-a}\int_0^x\ldots.\int_0^x}_{\longleftarrow n\longrightarrow} e^{-x}dx\ldots.dx+$$

Remainder becomes

$$R_2=[f^{(n+1)}(a+\Theta h)+f^{(n)}(a+\Theta h)]\underbrace{\int_0^{x-a}\int_0^x\ldots.\int_0^x}_{\longleftarrow n+1\longrightarrow} e^{-x}dx\ldots.dx$$

Adding equations (1) and (4) and dividing by 2

$$f(x)=f(0)\frac{e^x+e^{-x}}{2}+f'(0)\int_0^x\frac{e^x+e^{-x}}{2}dx-f(0)\int_0^x\frac{e^x-e^{-x}}{2}dx$$
$$+f''(0)\int_0^x\int_0^x\frac{e^x+e^{-x}}{2}dxdx-f'(0)\int_0^x\int_0^x\frac{e^x-e^{-x}}{2}dxdx$$
$$+f'''(0)\int_0^x\int_0^x\int_0^x\frac{e^x+e^{-x}}{2}dxdxdx$$

$$-f''(0)\int_0^x\int_0^x\int_0^x \frac{e^x-e^{-x}}{2}dxdxdx+\ldots.$$

or

$$f(x)=f(0)\cosh x+f'(0)\int_0^x \cosh x dx-f(0)\int_0^x \sinh x dx$$

$$+f''(0)\int_0^x\int_0^x \cosh x dxdx-f'(0)\int_0^x\int_0^x \sinh x dxdx$$

$$+f'''(0)\int_0^x\int_0^x\int_0^x \cosh x dxdxdx$$

$$-f''(0)\int_0^x\int_0^x\int_0^x \sinh x dxdxdx+\ldots.$$

$$+f^{(n)}(0)\underbrace{\int_0^x\ldots.\int_0^x}_{\longleftarrow n\longrightarrow}\cosh x dx\ldots.dx$$

$$-f^{(n-1)}(0)\underbrace{\int_0^x\ldots.\int_0^x}_{\longleftarrow n\longrightarrow}\sinh x dx\ldots.dx+\ldots.$$

As examples of some particular expansions

let $f(x)=x.$ $;f(0)=0$

Then $f'(x)=1$ $;f'(0)=1$

$f''(x)=0$ $f''(0)=0$

$$\therefore f(x)=x=\int_0^x e^x dx-\int_0^x\int_0^x e^x dxdx,$$

or $$f(x)=x=\int_0^x e^{-x}dx+\int_0^x\int_0^x e^{-x}dxdx,$$

or $$f(x)=x=\int_0^x \cosh\, x dx-\int_0^x\int_0^x \sinh\, x dxdx.$$

Let $f(x)=1$

$$f(0)=1;\ f'(0)=f''(0)=\ldots.f^{(n)}(0)=0$$

$$\therefore f(x)=1=e^x-\int_0^x e^x dx$$

or $$f(x)=1=e^{-x}+\int_0^x e^{-x}dx$$

or $$f(x) = 1 = \cosh x - \int_0^x \sinh x dx$$

Let $f(x) = x^n$

$$f'(0) = f'(0) = \ldots . f^{(n-1)}(0) = 0$$

$$f^{(n)}(0) = n!$$

$$\therefore f(x) = x^n = n! \underbrace{\int_0^x \ldots . \int_0^x}_{\leftarrow n \rightarrow} e^x dx \ldots . dx - n! \underbrace{\int_0^x \ldots . \int_0^x}_{\leftarrow n+1 \rightarrow} e^x dx \ldots . dx.$$

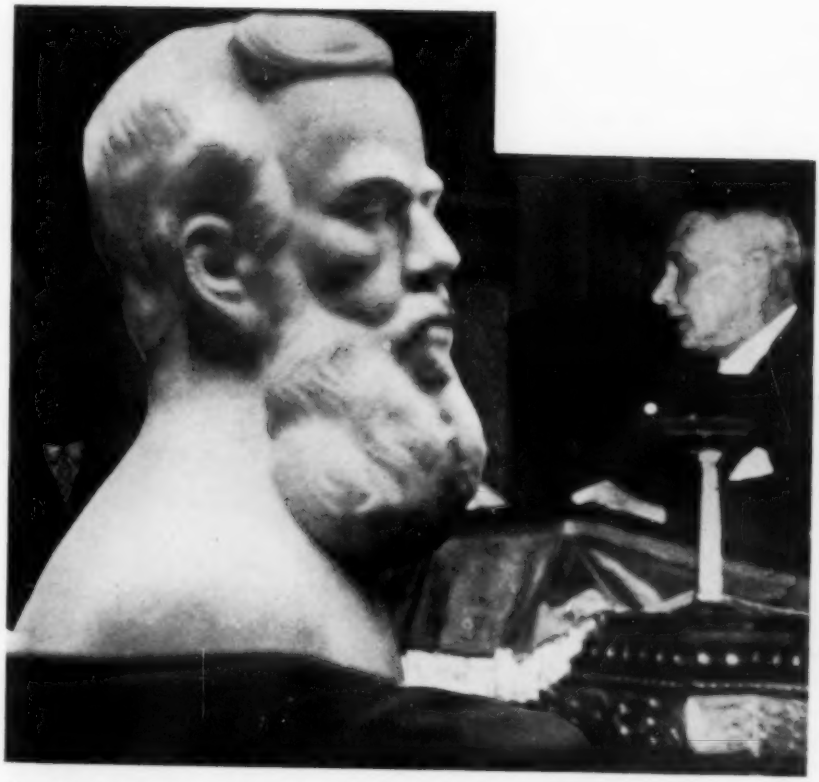
$$\text{or } f(x) = x^n = n! \underbrace{\int_0^x \ldots . \int_0^x}_{\leftarrow n \rightarrow} e^{-x} dx \ldots . dx + n! \underbrace{\int_0^x \ldots . \int_0^x}_{\leftarrow n+1 \rightarrow} e^{-x} dx \ldots . dx.$$

or

$$f(x) = x^n = n! \underbrace{\int_0^x \ldots \int_0^x}_{\leftarrow n \rightarrow} \cosh x dx \ldots dx - n! \underbrace{\int_0^x \ldots \int_0^x}_{\leftarrow n+1 \rightarrow} \sinh x dx \ldots dx.$$

Expansions in terms of integrals of $e^{ix}$ and $e^{-ix}$ may be obtained by the same methods as used above. Combining them gives expansions in terms of integrals of sin $x$ and cos $x$ similar to the above expansions in terms of sinh $x$ and cosh $x$.

Opening session, International Congress of Mathematicians, Aula, University of Oslo. July 14, 1936. King Haakon VII is seated in the center aisle in a chair to the left.



Georg Lous, Oslo Lawyer, presenting the bust of Sophus Lie to the University of Oslo, July 15th, 1936, at the session of the International Congress of Mathematicians.

Dean Poul Heegaard, University of Oslo, accepting the Sophus Lie bust in the name of the University.
Aula, July 15, 1936. International Congress of Mathematicians.

## Humanism and History of Mathematics

*Edited by*
G. WALDO DUNNINGTON

# *Oslo Under the Integral Sign*

By G. WALDO DUNNINGTON

Norway and Switzerland are not large countries, but their significance in the history of mathematics is not difficult to assess. One need merely mention for the former, Abel and Sophus Lie; for the latter, Euler and the Bernoulli family. The choice of Oslo as a meeting place for the International Congress of Mathematicians, July 13-18, 1936, proved to be a most happy and satisfactory one. Harald the Hard Hearted, one of the Viking kings, founded Oslo in 1045, and Haakon V built about 1300 a typical medieval fortress or castle called Akershus on a rocky cliff on the Oslo fjord. Less than 300 years ago King Christian IV founded a new city on a rocky plateau below Akershus and called it Christiania. In 1925 the name Oslo was restored, since it had been thus known from 1047 to 1624. Greater Oslo has today a population of about 400,000 inhabitants, a pleasing climate, is clean, and as a result of all this, is visited by many tourists. The museums are at once ideal and unique. Probably the Viking ships interested the mathematicians more than any other exhibit. The writer of these lines cannot resist recording here a delightful dinner and an evening spent with Dr. Dietrich Hildisch, the Consul General, and Mrs. Hildisch, at their lovely island estate in the Oslo fjord.

The meetings of the Congress were held in the buildings of the University of Oslo, which were erected during the years 1839-1854. On Monday evening, July 13, Prof. Sem Saeland, rector of the University, gave a reception for visiting mathematicians and their families in the "Aula". He was represented by Prof. Poul Heegaard, dean of the faculty of mathematics and natural sciences, who spoke warm words of welcome in French, German, and English. This was an evening of social pleasure and reunion for the mathematicians, in the beautiful Aula which is adorned with the colossal murals of Edvard Munch. In front of the Aula are monuments of Schweigaard and P. A. Munch, two important historians. On the campus of the University are several rune stones, one of which is about 1500 years old,

Several other important, interesting buildings which space forbids us to describe here, are: The Storting (parliament), the National Theater, the Palace of Justice, the Nobel Institute, the University library with 800,000 volumes, the Stock Exchange, and the Navigation School. Oslo has been called the "City of Frescoes", since it probably surpasses other European cities in this respect.

The formal opening session of the Congress occurred on Tuesday morning, July 14, in the Aula, King Haakon VII being present. Prof. Carl Størmer, astrophysicist at Oslo University and chairman of the Committee on Organization, officially opened the Congress. Official greetings were extended to the Congress by the Norwegian Minister of Foreign Affairs. Prof. Størmer was then elected President of the Congress. This was a well deserved honor, since he seems to have done more than any other individual to make it a success. In this connection it should be stated that Prof. Alf Guldberg, who had served as chairman of the above committee, died on February 15, 1936. The Fields Medals, established in 1932 at the Zurich Congress, were conferred on Prof. Lars V. Ahlfors, of Helsingfors, Finland, and Prof. Jesse Douglas, of Cambridge, Mass., for their work in function theory. Prof. Elie Joseph Cartan, University of Paris, officiated at this event.

Prof. Størmer lectured at the above session on a "*Program for the Quantitative Discussion of Electron Orbits in the Field of a magnetic Dipole, with Application to Cosmic Rays and Kindred Phenomena*". This lecture was unusually well illustrated on the screen. Following this, Prof. R. Fueter (Zurich) gave a paper entitled "*Die Theorie der regulären Funktionen einer Quarternionenvariablen*". At the conclusion of this morning session the delegates and participants were photographed in front of the Aula.

The afternoon of July 14th was given over to sectional meetings, which will be touched on later. At 5:30 p. m. the Congress members were received by King Haakon and Queen Maud at tea in the Royal Castle. This castle was built in 1848 and although somewhat larger, reminds one in external appearance of the White House. It is situated on a hill at the end of Karl Johann-gate, Oslo's main street, just beyond the University, and commands a splendid view of the city. The castle is surrounded by a stately park; in front of it is an equestrian monument of King Karl Johann. In the park are a portrait bust of Camilla Collett, the pioneer Norwegian feminist, and a monument of Abel, both by the great Norwegian sculptor Gustav Vigeland. The writer plans to publish some brief discussion of this Abel monument in the next issue of the NATIONAL MATHEMATICS MAGAZINE. Nearby is Vigeland's statue of the composer Richard

Nordraak, who wrote Norway's national hymn "Ja, vi elsker". The rather democratic royal family in Norway is extremely popular. Besides the royal couple, it consists of Crown Prince Olaf, who acted as honorary president of the Congress, Crown Princess Martha, and their two little daughters, the Princesses Ragnhild and Astrid.

From an historical viewpoint the general session on Wednesday morning, July 15, was probably the high watermark of the Congress. Mathematicians from 38 nations were assembled in the Aula for the unveiling of Dyre Vaas' bust of Sophus Lie. According to the program, Director J. Sejersted Bødtker was to speak and present the bust to the University of Oslo. However, he sent a telegram stating that he was unavoidably detained by a late train, and thus Georg Lous, an Oslo lawyer, spoke as follows:

"In the name of the friends of the Sophus Lie family and also of the admirers of his mathematical genius, I have the honor of presenting to our university a bust of this our great mathematician. Beside Niels Henrik Abel's name that of Sophus Lie is one of those which shines brightest in our mathematical firmament, indeed in all Norwegian science. As far as I know this will not be the first work of art by one of our distinguished sculptors to ornament our University on Blindern (hill). In asking that the veil be dropped, I desire to express the hope that the University will cherish this work of art, and that in the future on Blindern (hill) many busts of great Norwegian scientists will be erected, who shall have cast as much splendor over science and their fatherland as that potent Norwegian genius, Sophus Lie."

Dean Poul Heegaard accepted the gift in the name of the University. He expressed the hope that the mathematical work in the University of Oslo would always be worthy of this great model, now embodied and symbolized by the bust. He felt it a good omen that this hope may be fulfilled by the fact that Prof. Cartan was present to give the historical lecture (interspersed with personal reminiscences) entitled: "*The rôle of Sophus Lie's theory of groups in the development of modern geometry*". (Incidentally, it may be mentioned that Prof. Cartan early in September gave a paper at the Harvard Tercentenary Conference on "*The Extension of Tensor Analysis to Non-Affine Geometries.*")

Dean Heegaard said: "I am glad to be able to state on this occasion that the printing of Lie's Collected Works, with the exception of a fascicle of notes, is now completed. For this we are grateful to all institutions and persons who have helped us; especially to the Norwegian Mathematicians Association and above all to the untiring

efforts of Prof. Friedrich Engel (University of Giessen). The Works have become not only a monument to Sophus Lie, but also to him."

Prof. Engel, a pupil of Lie, was prevented at the last moment from attending the Congress, which sent him a telegram of greetings and admiration. Following this, two lectures were given simultaneously: Prof. Carl Ludwig Siegel (Frankfort a. M.) on "*Analytic Theory of Quadratic Forms*" concerning which we plan to report more in detail in a later issue, and Prof. Oswald Veblen (Princeton) on "*Spinors and Projective Geometry.*" Prof. Jakob Nielsen (Copenhagen) then gave a paper on "*Topologie der Flächenabbildungen.*" Unfortunately an important paper "*Théorie des nombres transcendants*" by Prof. A. O. Gelfond (Moscow) had to be omitted since he and the other Russian delegates did not appear. No one seemed to know the exact reason for this. The writer was told that Italy was not represented since Norway had belonged to the sanctions.

The afternoon was devoted to sectional meetings and a session of the International Commission on the Teaching of Mathematics. This session was presided over by Prof. Henri Fehr (Geneva); unfortunately the reports read from the various countries were rather dry and perfunctory, and there was little enthusiasm or discussion. However, the report of Dr. Walter Lietzmann, professor in Göttingen and vice-chairman of the Commission, was exceedingly interesting and enlightening as to mathematics in the schools of present-day Germany. Much praise is also due the Japanese delegates who illustrated their report with actual textbooks used there, and distributed to those present a well printed summary in pamphlet form. We hope to review this in our Teachers' Department in an early issue.

The evening of July 15th the city of Oslo tendered a banquet at the Hotel Bristol to the members of the Congress. Several prominent city officials and mathematicians present spoke briefly after the dinner, followed by a social evening in the upper parlors of the hotel.

A genuine master in his field, Prof. Ernst Hecke, University of Hamburg, started off the Thursday morning session (July 16th) with a paper "*New Progress in Theory of Elliptic Modular Functions.*" Dr. Otto Neugebauer (Copenhagen) furnished a highlight on the program with his well illustrated lecture "*On pre-Grecian Mathematics and its Comparison with the Grecian,*" stressing always the algebraic nature of the former and the geometric nature of the latter. Following this, Prof. C. W. Oseen (Stockholm) lectured on "*Problems in Geometric Optics*" and Prof. V. Bjerknes (Astrophysical Institute, Oslo; president of the Norwegian Academy of Science and Letters, son of C. A. Bjerknes, Abel's biographer) gave a paper on "*New Lines in Hydrodynamics.*"

Prof. Helmut Hasse, head of the mathematics department in the University of Göttingen, closed the morning session with a brilliant lecture "*Über die Riemannsche Vermutung in Funktionenkörpern.*"

Thursday afternoon and evening, July 16th, were devoted to a trip through the Oslo fjord on the SS. *Stavangerfjord*, the largest vessel of the Norwegian American line. The Crown Prince and Princess participated in this excursion, and at 6 o'clock a banquet was served in four dining halls of the ship. Following the dinner Prof. Størmer gave the opening address; the secretary-general of the Congress, Prof. Edgar B. Schieldrop (Oslo) welcomed the guests. The last named deserves special praise for the splendid and efficient manner in which he handled the entire secretariat of the Congress, in the Domus Academica of the University. The response to the above welcome was given by Prof. Emile Borel (Paris). These brief addresses were transmitted to all the dining halls by loudspeaker, and were interspersed with music of Grieg, Sibelius, Strauss, and a march dedicated to Crown Prince Olaf. In the evening there was dancing and cardplaying, happy conversation and reminiscing, with restaurant and bar on board in full swing. The ship docked in Oslo at midnight and taxis were waiting to take the guests to their hotels.

Dean G. D. Birkhoff (Harvard) opened the Friday morning session with a lecture "*On the Foundations of Quantum Mechanics,*" going back to and elaborating certain ideas he had worked out as early as 1907. He plans to publish his final results soon in a memoir in the Proceedings of the National Academy of Arts and Sciences. He was specially honored by the presence of Crown Prince Olaf at his lecture, and had the further pleasure of seeing his son, Garrett Birkhoff give papers in three sections of the Oslo Congress on "*Generalized Convergence,*" "*Order and the Inclusion Relation,*" and "*Product Integration of Non-linear Differential Equations.*" Mrs. Birkhoff accompanied him; they attended the 550th anniversary Jubilee of Heidelberg University, the Centenary of the University of London, and returned home to prepare for the Harvard Tercentenary Mathematical Colloquium.

Four additional lectures featured the Friday morning general session. as follows: Prof. S. Banach (Lwow, Poland) on "*Le rôle de la théorie des operations dans l'analyse*"; Prof. L. J. Mordell (Manchester, England) on "*Minkowski's Theorems and Hypotheses on Linear Forms*"; Prof. Lars V. Ahlfors (Helsingfors) on the "*Geometry of Riemann Surfaces*"; and Prof. J. G. van der Corput (Groningen) on "*Diophantine Approximations.*" Sectional meetings occupied the afternoon.

On Saturday morning (July 18th) the lecture of Prof. A. Khintchine (Moscow) on "*Main Features of the Modern Theory of Probability*" had to be omitted, due to his absence. There were three remaining lectures, as follows: Prof. Maurice Fréchet (Paris) on "*Mélanges mathématiques*"; Prof. Norbert Wiener (Cambridge, Mass.) on "*Tauberian Gap Theorems*"; and Prof. Øystein Ore (Yale) on "*The Structure of Algebraic Systems.*" There were sectional meetings during the afternoon.

The closing session of the Congress convened in the Aula at 5:15 p. m. At this session a telegram of thanks (replying to greetings sent him) from Prof. Friedrich Engel was read. Letters of thanks from the Queen and Crown Princess for flowers the Congress had sent them, were read. The chairman read telegrams of greeting and admiration sent to David Hilbert in Göttingen, Emile Picard in Paris, and Vito Volterra in Rome. Prof. Henri Fehr (Geneva) then reported on the session of the International Commission on the Teaching of Mathematics. Prof. Gaston Julia (Versailles) spoke about the International Mathematical Union.

Dean Luther P. Eisenhart (Princeton) presented the invitation of the American mathematical organizations to hold the next International Congress of Mathematicians (1940) in New York or a smaller city on the Atlantic seaboard. This invitation was accepted with loud, prolonged applause. Prof. J. A. Schouten (Delft) gave the farewell address for the guests. He thanked most heartily Norway, the city of Oslo, the Committee on Organization, and the Committee for Entertainment of the Ladies. Prof. Størmer then expressed congratulations to those present, best wishes for the coming years, and closed the Congress.

* * * * * *

The Congress was divided into eight sections, as follows:

I. Algebra and Number Theory
II. Analysis.
III. Geometry and Topology.
IV. Calculus of Probabilities, Mathematical Statistics, Actuarial Science, etc.
V. Mathematical Physics, Astronomy, and Geophysics.
VI. Mechanics and Engineering Sciences.
VII. Logic, Philosophy, and History.
VIII. Pedagogy.

Section II had three subsections, and Section III had two subsections. A total of forty sectional meetings took place; papers were limited to fifteen minutes, with the result that 227 were given. In addition to this, there were last-minute supplementary papers in the several sections, plus discussion of the papers. Papers in the sections were thus distributed:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| I. | 27 | III. | 52 | V. | 24 | VII. | 13 |
| II. | 79 | IV. | 23 | VI. | 15 | VIII. | 3 |

Abstracts of every paper presented, covering a maximum of one printed page each, will appear in the official Proceedings of the Congress. Three outstanding papers in section I were: Prof. Hermann Weyl (Princeton) on "*Factor Systems and Riemann Matrices*", Dr. Kurt Mahler (Krefeld a. Rh.), brilliant young pupil of Siegel, on "*Pseudo-evaluations,*" and Dr. N. Hofreiter (Vienna) on "*Approximation of Complex Numbers.*" Abstracts of papers by Tchakaloff and Köthe (section II) are to be found appended to this article.* In section III Prof. Harald Geppert (Giessen) spoke on "*Über den gemischten Inhalt zweier Bereiche*" and Prof. Louis Locher ((Winterthur) on the "*Structure of the Axioms of Projective Geometry.*" Section IV was deprived of a valuable paper "*On Unified Nomenclature in Mathematical Statistics*" by Prof. Wilhelm Lorey (Frankfort a. M.), well known historian of mathematics, who was prevented from attending. Dr. Fritz Noether (Tomsk, Siberia), brother of the late Emmy Noether, gave a paper in section V "*Über elektrische Drahtwellen.*" Father Georges Lemaître (University of Louvain), the internationally famous astrophysicist, gave a paper in section V entitled "*Results of calculations of asymptotic trajectories in the field of a magnetic dipole with applications to cosmic radiation.*" Prof. Rolf Nevanlinna (Helsingfors) who had just received the honorary doctorate at the 550th anniversary Jubilee of Heidelberg University for his work in function theory, was present at the Congress. Section VII, devoted to the history of mathematics, held two meetings. A paper by Locher on "*Goethe's Attitude to Mathematics*" will appear in the December issue of the NATIONAL MATHEMATICS MAGAZINE. Prof. Otto Spiess (Basel) made the following report:

"Jacques and Jean Bernoulli were for a long time the only mathematicians who completely comprehended the infinitesimal calculus created by Leibniz. In constant correspondence with the founder they developed, frequently in opposition to the Newtonian school, the fundamental methods of higher analysis. Jean Bernoulli's hun-

*These will appear in the December issue.—S. T. S.

dreds of letters, exchanged with the most eminent mathematicians of his time on this subject, constitute a source of the first order for the history of science and a valuable record of the mathematicians themselves. By force of circumstances this source has remained almost entirely unused up to the present, except that Jean Bernoulli's correspondence with Leibniz appeared in 1745. A century later the correspondence of the Bernoullis with the St. Petersburg circle (Euler, Fuss, and Goldbach) appeared. But the greater portion, about 2,000 letters, remained in oblivion until the Swiss astronomer and biographer R. Wolf solved the mystery after several decades of effort. He discovered that all the scientific correspondence of Jean Bernoulli, as well as the equally important letters of the other Bernoullis, was sold just before 1800 by a grandson of Jean, partly to the Academy at Stockholm and partly to the Prince of Gotha. This correspondence lay hidden and no use was made of it, either by the Prince or the Academy. Even since its discovery, only a small circle has known of its existence. Dr. Rudolf Geigy, a wealthy Basel industrialist, has donated to the Basel Society of Natural Sciences a sum of 30,000 francs and thus created the Bernoulli Foundation."

The publication of these letters has been entrusted to Prof. Spiess, who has selected some editorial collaborators; the Swiss Mathematical Society is also interested in the publication of these letters, which will likely require a number of years, because of their voluminous nature. Anyone having Bernoulli original letters or copies (to or from them) or other pertinent information, is urged to write to Prof. Spiess. It is hoped that as many institutions and individuals as possible will subscribe to this series.

The second session of the historical section was opened by Prof. R. C. Archibald (Brown University) who presented interesting material supplementary to his Sylvester article in Osiris (January, 1936); he gathered much of this material on his recent travels in Europe. Prof. Solomon Gandz (New York) gave a paper on the "*Invention of the decimal fractions and the exposition of the exponential calculus by Immanuel Bonfils, circa 1350.*" The paper by Dr. A. N. Singh (Lucknow) on the history of magic-squares in India shows how many treasures in the way of old mathematical manuscripts are lying about there, unedited and unknown to most scholars. Dean Heegaard's brief description of some numbers found in an old papyrus fragment in the Oslo collection precipitated a lively discussion, which was participated in by Dr. Neugebauer, the leading figure in this field. Dr. József Jelitai (Budapest) gave a brief sketch of the history of mathematics in Hungary. Dr. Kurt Vogel (Munich) spoke on Babylonian mathe-

matics, and the author of this report closed the section with a brief statement about several Gauss publications planned in Göttingen and America. Prof. Otto Spiess presided over this session.

* * * * * *

In closing this report special mention should be made of the excellent work of the Committee on the Entertainment of Ladies, which included the widow of Prof. Guldberg, and the wives of Professors Heegaard, Schieldrop, Skolem, and Størmer. This entertainment included sightseeing by bus, side trips to Frognerseteren and Skaret, visiting the museums and art galleries, and a tea given by Mrs. Henny Løvenskiold on her estate at Baerum's Verk.

The information office at the Congress secretariat in the Domus Academica and the writing room there were well managed. The mail of the guests was well handled, and the students of the University were very helpful to the guests in various incidental matters. Delegates to the Congress wore a badge in the form of an integral sign, which entitled them to ride free on the street cars and busses in Oslo and vicinity. The book stores featured special exhibits of mathematical books and materials during the week. Steamship lines and railroads serving Norway allowed special rates to the visiting mathematicians, many of whom took advantage of this to remain a week after the Congress, in order to see beautiful scenery for which Norway is famous.

## STATISTICS ON THE OSLO CONGRESS

| *Nation* | *Delegates* | *General Lectures* | *Sectional Papers** | *Sectional Distribution of Papers* |
|---|---|---|---|---|
| U. S. A. | 81 | 4 | 28 | I-VIII |
| Norway | 59 | 2 | 13 | I, II, IV, V, VI, VII |
| England | 41 | 1 | 21 | I-V |
| Sweden | 28 | 1 | 13 | I, II, IV |
| France | 28 | 2 | 15 | I-VII |
| Poland | 24 | 1 | 16 | II, III |
| Denmark | 23 | 2 | 2 | III, VI |
| Germany | 20 | 3 | 13 | II, III, IV, VI, VII |
| Switzerland | 20 | 1 | 11 | I-IV, VI, VII |
| Holland | 16 | 1 | 9 | I, III, V |
| Czecho-Slovakia | 11 | .. | 8 | I, II, III, V |
| Rumania | 10 | .. | 10 | II-VI |
| Austria | 10 | .. | 8 | I-IV |
| Finland | 9 | 1 | 5 | II, V |
| Scotland | 9 | .. | 1 | V |
| Belgium | 9 | .. | 8 | I-III, V-VII |
| Ireland | 8 | .. | 2 | II, V |
| Canada | 8 | .. | 2 | III, V |
| Hungary | 7 | .. | 5 | I, III, VII |
| Spain | 7 | .. | 3 | II, III |
| Japan | 5 | .. | 1 | I |
| Greece | 5 | .. | 1 | III |
| Egypt | 4 | .. | 3 | III |
| Palestine | 4 | .. | 3 | II, III |
| Bulgaria | 3 | .. | 3 | II |
| Siberia | 3 | .. | 1 | V |
| Jugo-Slavia | 3 | .. | 2 | VI |
| India | 3 | .. | 3 | II, VII |
| Persia | 2 | .. | 1 | V |
| South Africa | 2 | .. | .. | ..... |
| Latvia | 2 | .. | .. | ..... |
| Argentina | 1 | .. | .. | ..... |
| Esthonia | 1 | .. | .. | ..... |
| Iceland | 1 | .. | 1 | II |
| Danzig | 1 | .. | .. | ..... |
| China | 1 | .. | .. | ..... |
| Australia | 1 | .. | .. | ..... |
| Algeria | 1 | .. | 1 | IV |
| Totals | 424 | 19 | 227 | |

*(Note). Russia was listed for 24 papers in sections I-VI, Italy for two in II. The number of papers each nation contributed to each section might also prove interesting. The total number of delegates does not include wives, children, and others accompanying delegates.

## The Teacher's Department

*Edited by*
JOSEPH SEIDLIN

# *The Use of Matrices in College Mathematics*

By H. T. R. AUDE
*Colgate University*

The notation and use of matrices may well be introduced in college mathematics in conjunction with certain topics in algebra. But in analytic geometry there is found an especially favorable place for their early use. In this paper various examples from the latter field will be selected to illustrate certain advantages which are outstanding when matrices are used, not only as symbols of notation, but also as operators. Such advantages are probably best shown by actual use. For applications to problems carry in their favor not only that of "learning by doing," but also help to an appreciation of a new tool. Some of the operations with matrices will be found to give results in a simple and direct manner. Their place in elementary mathematics will be justified when their use leads to results in an efficient and interesting manner.

Even though it may seem somewhat like "text and study" procedure it is urged that the reader—to whom this is not familiar ground—follow through the subsequent problems with pencil and paper, not only to check upon the statements, but also to verify some of the results by using other methods. Next, it is hoped that the reader will apply these methods of operating with matrices to some examples of his own selection. And, finally, if it may have found favor so far, to have students try it in the classroom.

Among the examples selected to show the use of matrices are the following:

The matrix for a point and for a line.

Any point on the line determined by two given points, and its dual.

The matrix notation for the second degree equation in the plane. Polarization by matrices; tangents; pole and polar.

The quadric surfaces.

The quadratic form in one dimension. The harmonic of a point with respect to two other points which may represent numbers, real or complex.

## MATRICES FOR POINTS AND LINES

The matrix for the point whose Cartesian coordinates are $(x,y)$ will be $(x\ y\ 1)$. This can be regarded as a notation from homogeneous coordinates. The forms $(x\ y\ 1)$, $(2x\ 2y\ 2)$ $(rx\ ry\ r)$ are equivalent. Thus the point matrix $(a\ b\ c)$ denotes the point whose Cartesian coordinates are $(a/c,\ b/c)$.

This notation that the abscissa is $x/1$ and the ordinate is $y/1$ is in harmony with the best definitions of abscissa and ordinate. It allows us to state that abscissas and ordinates are numbers not distances, not denominate or dimensional numbers, just numbers.

The matrix for the linear equation $ax+by+c=0$ will be the line matrix $(a\ b\ c)$. To avoid confusion between the matrices for points and lines in the plane it will be best to state—whenever necessary—the element, point or line, which the matrix represents.

The points whose matrices are $P_1(2-3\ 1)$, $P_2(5\ 2\ 1)$, $P_3(7-1\ 2)$ all lie on the line whose matrix is $p(5\ -3\ -19)$.

A matrix is a rectangular array of numbers. When suitable, or necessary, a matrix as

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

may also be written in the transposed form

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

This second form is needed so that the multiplication of matrices may proceed by the rule—"row by column"—where the sum of the products of the elements of the row $h$ by the corresponding elements of the column $k$ gives the element in row $h$ and column $k$ of the resulting matrix. Thus the matrix of a line $p$ may be multiplied by the matrix of a point $P$. The matrix of $p$ is then followed on its right by the transposed matrix of $P$. Choose the line $p$ $ax+by+c=0$ and the point $P(x_1,\ y_1)$ then the multiplication is given by the equation

$$p \cdot P = (a\ b\ c)\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = ax_1+by_1+c.$$

It may be of interest to note that when this product is zero there is united position of the point and the line; the point $P$ lies on the line $p$.

The two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ have the matrices $P_1(x_1\ y_1\ 1)$ and $P_2(x_2\ y_2\ 1)$. If the second matrix is multiplied by $\tau$ and added to the first there results the matrix of the point

$$P(x_1+\tau x_2\ y_1+\tau y_2\ 1+\tau)$$

whose Cartesian coordinates are

$$(1) \qquad x=\frac{x_1+\tau x_2}{1+\tau} \qquad y=\frac{y_1+\tau y_2}{1+\tau}$$

This point $P$ lies on the line determined by the points $P_1$ and $P_2$ for every value of $\tau$. There is no exception for if $\tau=-1$ there results the point at infinity on the line through $P_1$ and $P_2$ in the direction of

$$\tan^{-1}\frac{y_2-y_1}{x_2-x_1}.$$

When the matrices for the points are written in the form such that the third element is 1, thus $(x\ y\ 1)$, then *when $\tau$ in equations* (1) *is equal to 1 the point $P(x,y)$ is the mid-point of the line segment $P_1P_2$.*

Furthermore, the equations in (1) are a set of parametric equations of the straight line through the points $P_1$ and $P_2$. The parameter $\tau$ may be eliminated between these and there results the equation

$$\frac{y-y_1}{x-x_1}=\frac{y-y_2}{x-x_2}$$

A better way, however, of finding the equation of the line through the points $P_1$ and $P_2$ is to join the two matrices given above into the two-by-three matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}$$

Next, find the values of the three second order determinants formed in turn from columns two and three, three and one, one and two. These three numbers form the one rowed matrix of three elements which is the matrix for the line. These second order determinants are

$$\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

The above results are more quickly obtained by adding on the right the columns one and two, thus

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix}$$

From this array it is easy to write the values of the three second order determinants formed from the columns two and three, three and four, four and five:

$$y_1-y_2, \quad x_2-x_1, \quad x_1y_2-x_2y_1.$$

The matrix of the line $p$ through $P_1$ and $P_2$ is

$$(y_1-y_2 \quad x_2-x_1 \quad x_1y_2-x_2y_1),$$

from which the equation of $p$ is written

$$(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1=0.$$

For example, to find the equation of the line through the points $P_1(2,5)$ and $P_2(-4,3)$, there is first formed the matrix from the two points

$$\begin{pmatrix} 2 & 5 & 1 \\ -4 & 3 & 1 \end{pmatrix}$$

This is augmented by adding columns one and two on the right giving the array

$$\begin{pmatrix} 2 & 5 & 1 \\ -4 & 3 & 1 \end{pmatrix} \begin{matrix} 2 & 5 \\ -4 & 3 \end{matrix}$$

The values of the three second order determinants, beginning with columns two and three, are 2, $-6$, and 26. The matrix of the line $p$ through $P_1$ and $P_2$ is $(2 \ -6 \ 26)$, and the line $p$ has the equation $x-3y+13=0$.

The midpoint $P$ of the segment $P_1P_2$ is found by merely adding the two matrices $(2\ 5\ 1)$ and $(-4\ 3\ 1)$. The matrix of $P$ is therefore $(-2\ 8\ 2)$ and its coordinates are $x=-1$, $y=4$.

The two lines $p_1$ $a_1x+b_1y+c_1=0$ and $p_2$ $a_2x+b_2y+c_2=0$ have the matrices $p_1$ $(a_1\ b_1\ c_1)$ and $p_2$ $(a_2\ b_2\ c_2)$. If the second matrix is multiplied by $r$ and added to the first there results the matrix of the line $p$ $(a_1+ra_2\ b_1+rb_2\ c_1+rc_2)$ whose equation is $(a_1+ra_2)x+(b_1+rb_2)y+c_1+rc_2=0$. This line $p$ passes through the point determined by the two lines $p_1$ and $p_2$ for all values of $r$. Furthermore, by assigning

values to $r$ such that in turn the coefficients of $y$ and $x$ vanish there results

$$(2) \qquad x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \quad y=\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}.$$

A better way, however, for solving the two equations of $p_1$ and $p_2$ is to join the matrices of $p_1$ and $p_2$ into the two-by-three matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

Augmenting this, by adding on the right columns one and two, given the array

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2. \end{matrix}$$

The values of the three second order determinants formed from columns two and three, three and four, four and five, are

$$b_1c_2-b_2c_1, \quad a_2c_1-a_1c_2, \quad a_1b_2-a_2b_1.$$

Whence the matrix of the point $P=(p_1,p_2)$ is

$$(b_1c_2-b_2c_1 \quad a_2c_1-a_1c_2 \quad a_1b_2-a_2b_1)$$

and the coordinates of $P$ are easily read to be as given in (2).

As a numerical example, consider the two lines $p_1$ $2x+5y-7=0$ and $p_2$ $3x-y+2=0$. Their matrices are $p_1$ $(2\ 5-7)$ and $p_2$ $(3\ -1\ 2)$. If the second matrix is multiplied by $r$ and added to the first there results the matrix of the line $p$.

$$(2+3r \quad 5-r \quad -7+2r)$$

which represents a line $p$ through the intersection of lines $p_1$ and $p_2$ for all values of $r$.

The coordinates of the common point are readily found after the matrix formed from $p_1$ and $p_2$ has been augmented by adding columns one and two on the right giving the array

$$\begin{pmatrix} 2 & 5 & -7 \\ 3 & -1 & 2 \end{pmatrix} \begin{matrix} 2 & 5 \\ 3 & -1 \end{matrix}$$

The values of the three second order determinants beginning with column two are 3, $-25$, $-17$. Whence the matrix of the point $P$ common to $p_1$ and $p_2$ is $(3\ -25\ -17)$, and its coordinates are $x=-3/17$, $y=25/17$.

The two preceding examples show the duality which exists between points and lines in the plane. This interesting relationship is made more evident when matrices are used by the similarities in notation and method.

## MATRICES FOR THE EQUATIONS OF SECOND DEGREE IN THE PLANE

The matrix for the equation of the circle $x^2+y^2-4x+6y-12=0$ will be the symmetric matrix

$$C\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & -12 \end{pmatrix}$$

where the reader supplies the factors $x$, $y$, and 1 respectively to the rows one, two, and three, and to the columns one, two and three. Thus the number

1 in the first row and in the first column, denotes $1xx$
1 in the second row and in the second column, denotes $1yy$
0 in the first row and in the second column, denotes $0xy$
3 in the second row and in the third column, denotes $3y1$
3 in the third row and in the second column, denotes $31y$
etc.

The following equations of second degree in $x$ and $y$ are now to be written as symmetric matrices:

(a) $y^2-2x+y-6=0$;
(b) $2xy+x-4y+10=0$;
(c) $2x^2-xy-y^2+4x+2y=0$;
(d) $x^2+2xy+y^2+x+y-6=0$.

The matrices for the first three are:

$$\text{(a)}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & -6 \end{pmatrix}; \quad \text{(b)}\begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -2 \\ \frac{1}{2} & -2 & 10 \end{pmatrix}; \quad \text{(c)}\begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ -\frac{1}{2} & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Note, that the value of the third order determinant of the matrix (c) is zero. A matrix of this type is called singular. It means that the equation (c), which it represents, is reducible. In other words, the equation in (c) can be written as the product of two linear factors; and its graph is two lines. The same statement is also true for the equation given in (d).

Let $t$ denote the tangent to the second degree plane curve $S$ at the point $P$ on the curve. It can then be verified that the product of the matrices $S$ and $P$ yields the matrix of $t$.

$$t = S \cdot P$$

A few examples will show the process.

The point $P$ (5, 1) is on the circle $S$: $x^2+y^2-4x+6y-12=0$, then from the above equation

$$t = S \cdot P = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & -12 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = (3 \quad 4 \quad -19).$$

and the equation of the tangent $t$ is $3x+4y-19=0$.

Similarly the point $P(3, 3)$ is on the parabola $S$: $y^2-2x+y-6=0$ (previously given in (a)). By multiplication of the matrices $S$ and $P$ there is found the matrix $t$ $(-1 \quad 7/2 \quad -15/2)$; and the equation of the tangent at $P$ is $2x-7y+15=0$.

The point $P(-2, 1)$ is on the hyperbola $S$ given in (b). The equation of the tangent at $P$ is readily found.

The point $P(0,0)$ is on the locus the equation of which is given in (c); and the point (0, 2) will satisfy the equation (d). If the singular matrices in (c) and (d) be multiplied respectively by the matrix of a point $P$ on the locus, then the process will give one of the linear factors. Unless the point used is on both lines, in which case the indeterminate matrix (0 0 0) will appear.

## POLARIZATION BY MATRICES

The general equation of the second degree in two variables, when matrices are used, is preferably written in the form

$$S \quad a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{13}x+2a_{23}y+a_{33}=0$$

Its matrix is

$$S \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ where } a_{ij}=a_{ji}$$

For any point $P(x, y)$ with the matrix $(x \; y \; 1)$ the matric equation

$$P = S \cdot P$$

yields a one row—three column matrix $p$ where $p$ represents the polar of the pole $P$ with respect to the curve $S$. If $P$ lies on the curve then it is the special case of coincidence of pole and polar or tangency.

Consider the point $P(1, -4)$ and the parabola $S\ y^2-4x=0$, then the matrix equation $p=S\cdot P$ yields

$$p=\begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}=(-2\ -4\ -2).$$

Whence the equation of $p$ is $-2x-4y-2=0$ or $x+2y+1=0$. And this is the polar of the point $P(1, -4)$ as pole with respect to the given parabola. The tangents from $(1, -4)$ to the parabola cut the line $x+2y+1=0$ in the same points as the curve $y^2-4x=0$.

If the origin is the pole $P$, and if $S$ is the ellipse $x^2+2xy+4y^2-2x+4y-15=0$ then the polar $p$ is readily found to be $x-2y+15=0$.

If the polar $p$ $6x+3y+10=0$ is given along with the ellipse $S$ in the previous example, then the pole $P(x, y)$ of matrix $(x_1\ y_1\ k)$ may be found from the equation $p=S\cdot P$. Substitution gives

$$(6\ 3\ 10)=\begin{pmatrix} 1 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & -15 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ y_1 \\ k \end{pmatrix}$$

Performing the multiplication yields the three equations

$$x_1+y_1-k=6,$$

$$x_1+4y_1+2k=3,$$

$$-x_1+2y_1-15k=10.$$

Solving these yields $x_1=5$, $y_1=0$, and $k=-1$. Whence the coordinates for $P$ are $x=-5$, $y=0$.

This could also be accomplished by multiplying the adjoint matrix $S'$ by the matrix $p$, then $P=S'\cdot p$. The adjoint $S'$ of the matrix $S$ is formed by replacing the element in row $h$ and column $k$ of the determinant $S$ by the cofactor of the element in row $k$ and column $h$.

## QUADRIC SURFACES

This method of polarization is applicable in a corresponding manner in the geometry of three dimensions. The tangent plane $\pi$ to a quadric $Q$ at a point $P$, or the polar plane of a point with respect to a quadric, according as the given point is or is not located on the quadric surface, are found from the matrix equation

$$\pi=Q\cdot P$$

For example, consider the hyperbolic paraboloid $Q\ x^2-4y^2-2z=0$. The point $P_1$ (4, 1, 6) is on its surface while the point $P_2$ (4, 1, 10) is not. The matrix for $Q$ is a four by four array of numbers where the coefficients of the third variable $z$ are assigned to the third place in the arrangement of rows and columns. Substituting the matrices in the equation, $\pi = Q \cdot P$ gives

$$\pi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 6 \\ 1 \end{pmatrix}.$$

Performing the multiplication yields for $\pi_1$, the matrix

$$\pi_1 = (4\ \ -4\ \ -1\ \ -6).$$

and the equation of the tangent plane is $4x-4y-z-6=0$. By using the second point $P_2$ in a similar manner there is obtained for the plane $\pi_2$ the equation $4x-4y-z-10=0$, which is the polar plane of $P_2$ with respect to the surface $Q$.

A second example, briefly stated, is that the tangent plane to the cone $x^2+4y^2-z^2+4x-8y+6z-1=0$ at the point $(-5, 3, 8)$ is $3x-8y+5z-1=0$.

## THE ONE-DIMENSIONAL QUADRATIC FORM

The preceding examples have shown how the problem of pole and polar to a second degree curve or surface may be handled in two or three dimensions, respectively. It is also interesting to note that when these methods are applied to the corresponding one-dimensional problem there results the fourth point $D$ which is the harmonic of $C$ with respect to the two points $A$ and $B$; which, of course, is a logical consequence.

For example, if $A$ is located on a line 4 units from the zero point on the line, then $x=4$. And if $B$ and $C$ have the coordinates $x=-2$ and $x=6$, respectively. Then, the matrices for $A$, $B$ and $C$ are respectively (4 1), (−2 1), and (6 1). The quadratic form $q$ which represents $A$ and $B$ is $(x-4)(x+2)=x^2-2x-8=0$. It has the two by two matrix $q \begin{pmatrix} 1 & -1 \\ -1 & 8 \end{pmatrix}$.

The matrix equation

$$D = q \cdot C$$

yields

$$D = \begin{pmatrix} 1 & -1 \\ -1 & 8 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \end{pmatrix} = (5\ \ -14).$$

Whence the equation for $D$ is $5x-14=0$, or $x=14/5$. And $D$ is located on the line 14/5 units from the origin. It is easily verified that

$$\frac{AC}{BC}\cdot\frac{BD}{AD}=-1$$

which shows that $D$ is the harmonic of $C$ with respect to $A$ and $B$.

The correponding results for complex numbers are of sufficient interest to merit an example of illustration.

The quadratic form $q$ $x^2+2x+5=0$ represents the two complex numbers $A$ $-1+2i$ and $B$ $-1-2i$. The matrix for $q$ is

$$\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}.$$

If the number $C1+2i$ be selected then the matrix equation

$$D=q\cdot C$$

yields

$$D=\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}\quad\begin{pmatrix}1+2i\\ 1\end{pmatrix}$$

Multiplication gives for $D$ the matrix $(2+2i\ \ 6+2i)$. The equation for $D$ is

$$(2+2i)x+6+2i=0$$

whence

$$x=-2+i.$$

Represent graphically in the number plane the four numbers $A$ $-1+2i$, $B$ $-1-2i$, $C$ $1+2i$, and $D$ $-2+i$. It will be found that the four points lie on a circle and that it is true that

$$\frac{AC}{BC}\cdot\frac{BD}{AD}=-1.$$

Again $D$ is the fourth harmonic to A, B and C.

# Mathematical Notes

*Edited by*
L. J. ADAMS

Professor Edgar E. DeCou, head of the mathematics department in the University of Oregon, announces that Dr. Andrew F. Moursund has been promoted to the rank of associate professor, beginning this fall. Dr. Andrew Moursund enjoys a fine reputation as both research worker and teacher.

Professor Charles R. Sherer, head of the mathematics department in Texas Christian University, announces:

1. Miss Elizabeth Shelburne, instructor of mathematics, has been appointed Assistant Dean of Women at Texas Christian University. Miss Shelburne will remain in the mathematics department for part time teaching.

2. Mr. H. J. Jones has been appointed an instructor of mathematics at Texas Christian University. Mr. Jones had been teaching in the public schools at Wichita Falls, Texas.

Professor Cornelius Gouwens informs us of the death of Dr. Julia T. Colpitts, Associate Professor of Mathematics at Iowa State College. Dr. Colpitts died in England on her way home from the World Mathematics Congress in Oslo. She was a joint author, with Professor Maria M. Roberts, of a textbook on Analytic Geometry. Her mathematical interests lay in the theory of functions, and particularly in infinite series.

There were seventy-eight doctorates, with mathematics or mathematical physics as major subject, conferred during 1935 in the United States and Canada.

The Office of Education, United States Department of the Interior, publishes an Educational Directory which is divided into three parts:

Part I. State and County School Officers.
Part II. City School Officers
Part III. Colleges and Universities.

The parts are sold separately for five cents each.

A new journal of great interest to some mathematicians is the *Journal of Symbolic Logic*. The first issue is dated March, 1936. The

editors are Alonzo Church and C. H. Langford. The managing editor is C. A. Baylis. This journal is published quarterly by the Association for Symbolic Logic, Inc. The first volume is planned to include a complete bibliography of symbolic logic from Leibniz to the present time.

Colonel Malcolm C. Rorty, former President of the American Statistical Association, and a leader among statisticians, died on January 18, 1936.

The paper *Solution of Waring's Problem*, by Professor L. E. Dickson of the University of Chicago, is to be found on page 530 of the July, 1936, issue of the *American Journal of Mathematics*. The *American Journal of Mathematics* is published by The Johns Hopkins Press of Baltimore, Maryland.

An interesting example of the use of mathematics in economics is the book *Family Expenditure* by R. G. D. Allen and A. L. Bowley. This is an almost purely mathematical treatment of family budget, and is expressed in the notation of functions, and partial derivatives and total differentials. The book is published in London by P. S. King and Son, Ltd.

At the recent meeting of the Royal Academy in Brussels, Belgium, the following mathematical papers were presented:

1. *Some involutions belonging to the generalized Humbert surface.* L. Godeaux.
2. *Flecnodal surfaces of a ruled surface.* E. Anglade.
3. *Linear congruence of conics.* L. Derwidué.
4. *Study of the cubic surfaces which can possess Eckardt points.* B. Gambier.
5. *Problems of the limits relative to certain systems of partial differential equations.* L. Martin.
6. Remark on the note: *"The logical foundations of the theory of probabilities,"* by S. Avsitidysky.

Two research papers in mathematics were presented at the May 4, 1936 meeting of the Polish Academy of Science and Letters. They were:

1. *A property of certain generalized logarithmic potentials.* G. Girand.
2. *Quaternions, rotations in space of four dimensions, and the formula of Cayley.* J. Wevssenhoff and A. Bielecki.

This meeting was held at Cracow. At the June 8, 1936 meeting, G. Girand discussed *A generalization of logarithmic potentials of a double layer.*

The May 12, 1936 meeting of the Imperial Academy at Tokyo, Japan, included the presentation of the following mathematical papers:

1. *Theory of affinor transformations.* H. Hombu.
2. *Existence of a Galois field with a given p group.* T. Tannaka.
3. *The algebras over a field with a prime number characteristic.* T. Nakayama.

Volume I of the Collected Works of G. A. Miller, published by the University of Illinois, contains sixty-two articles written on various subjects by Professor Miller.

There will be a meeting of the American Mathematical Society at Lawrence, Kansas on November 27-28, 1936. By invitation of the program committee there will be two addresses:

1. *Fractions.* Professor L. R. Ford.
2. *Some topics in sampling theory.* Professor H. L. Rietz.

The Society will also meet in Los Angeles, California, on November 28, 1936.

A distinctive addition to the literature of mathematical recreations is the book *Recreations in Mathematics* by H. E. Licks, The puzzles are arranged by subjects, such as: algebra, arithmetic, geometry, physics, etc.

The July, 1936 (Part 2) issue of the *Bulletin of the American Mathematical Society* contains a general index for the years 1925-1934, inclusive. The index is arranged by authors and is divided into two sections. One section gives the list of research papers and the other is a list of reviews of mathematical and scientific books.

Mathematical research in Japan finds outlet through the following journals and publications:

1. *Journal of the Mathematical Association of Japan.*
2. *Journal of the Faculty of Science,* Imperial University of Tokyo.
3. *Journal of the Physics-Mathematical Society of Japan.*
4. *Memoirs of the College of Science,* Kyoto Imperial University.
5. *Memoirs of the Ryojun College of Engineering.*
6. *Proceedings of the Imperial Academy.*
7. *Proceedings of the Physics-Mathematical Society of Japan.*

8. *Science Reports of the Tôhoku Imperial University.*
9. *Tôhoku Mathematical Journal.*
10. *Journal of the Faculty of Science*, Hokkaidô Imperial University.
11. *Science Reports of the Tokyo University of Literature and Science.*
12. *Journal of Science of the Hiroshima University.*
13. *Memoirs of the Faculty of Science and Agriculture*, Taihoku Imperial University.
14. *Journal of the Society of Tropical Agriculture*, Taihoku Imperial University.

Professor D. C. Harkin, Alabama Polytechnic Institute, is giving a new senior course called Foundations of Mathematics. It is essentially an orientation course at the end of the undergraduate training, and tells the student what it was all about, rather than what it will be about. Historical and logical developments are stressed.

Professor R. D. Doner is the new head of the mathematics department at Alabama Polytechnic Institute. He takes the place of the late Professor B. H. Crenshaw.

Professor R. D. Doner, Alabama Polytechnic Institute, is the author of a paper, *The Dynamics of Granular Media*, in the July, 1936 issue of the journal *Agricultural Engineering*.

Mr. M. G. Moore has been appointed instructor in mathematics, Alabama Polytechnic Institute. He fills the vacancy created when Mr. A. C. Cohen accepted an appointment with the C. C. C.

# Book Reviews

*Edited by*
P. K. SMITH

*Introductory College Mathematics.* By F. E. Johnston. Farrar & Rinehart, Inc. New York, 1936, x+314 pages.

The author states in the preface "This book is designed for a four- or five-hour course given throughout the year." It is the opinion of the reviewer that the material included cannot be covered adequately in less time. Certain sections and chapters are designated as possible omissions in order to adapt the book to a three hour course. These include such topics as complex numbers, general theory of determinants, and the straight line in space.

The first two chapters are devoted to trigonometric functions and the relations connecting them. In chapter I sections on equations and identities and simplification of fractions are introduced just before the study of trigonometric identities. This method of treating algebra as a tool subject is followed throughout the book.

Chapter III is on logarithms and exponents. Negative, fractional, and zero exponents are defined. The logarithm, $l$, of a number, $N$, to the base, $b$, is defined to be such that the relations:

$$\log_b N = l,$$

$$b^l = N,$$

are equivalent. This definition is followed by a number of illustrations such as "$\log_3 81 = 4$, since $3^4 = 81$." The discussion of logarithms and the use of tables is very good and is sufficiently illustrated by examples.

Chapter IV, on the solution of triangles, gives the usual methods of solving general triangles by use of logarithms. In regard to right triangles, we find on page 66 "If the triangle is a right triangle, no special discussion is required, and all that is necessary is a knowledge of the trigonometric functions." There are very few problems on right triangles.

The usual forms for the equation of the straight are derived in chapter V. Determinants of order two and three are also introduced in this chapter.

The next three chapters are on certain topics in algebra. They include the quadratic equation with some of the theorems on polynomials, mathematical induction, the binomial theorem, and complex numbers.

Polar co-ordinates are introduced in chapter IX. Such topics as symmetry, extent, vertical and horizontal asymptotes and their help in sketching curves, are discussed.

The next four chapters are on the conic sections. The general equation of second degree and the reduction to standard form by transformation of axes are included. Differentiation is introduced in chapter XII and is used in determining the equations of tangents to the conics.

The last three chapters are on analytic geometry of three dimensions and determinants. Only the plane and the straight line are considered.

Answers to the exercises are given in the back of the book. The appendix includes four-place logarithms of numbers and of the trigonometric functions, the values of the trigonometric functions to four figures together with the radian measure of the angles, and a table of square roots of numbers from 1 to 250.

The discussions and illustrations, as a whole, are very good. There are some instances, however, where it seems that more familiar terms might have been used. We find on page 20 "$2x^\circ = 4$" as an example of an equation which has no solution. This occurs before the definition of the zero exponent, but the student is referred to the definition by a footnote. It seems that some other illustration might be better understood by the student. On page 111, in order to show that $\sqrt{2}$ is irrational, it is assumed that $\sqrt{2} = b/a$, where $a$ and $b$ are "relatively prime" integers. The student will very likely not know what the term means. On page 126 we have "...every equation of the form $f(x) = 0$, where $f(x)$ is a polynomial in $x$ of degree greater than or equal to one with coefficients of the form $a+bi$, $a$ and $b$ real, has a solution of the same form." Here the omission of a comma seems to alter the meaning of the statement. One feature of the text that will be appreciated by many is that "infinity" is not used in giving the values of the trigonometric functions. We find on page 12 "$\tan 270^\circ = -1/0 =$ (no such thing)." There are 116 figures many of them applying to examples worked out. There are exercises of sufficient variety under most of the topics.

*Louisiana State University.* W. V. PARKER.

*Analytic Geometry.* By John W. Young, Tomlinson Fort, and Frank M. Morgan. Houghton Mifflin Company, New York, 1936. x+347 pages.

The publishers call attention to the fact that "Analytic Geometry" by Young, Fort, and Morgan "originated as a posthumous and incomplete manuscript of the late John Wesley Young." Professor Fort and Professor Morgan were asked to complete the work begun by Professor Young. The two professors had been associated with Professor Young and were thoroughly conversant with Professor Young's method of instruction and his capacity for rigor without the sacrifice of simplicity.

There is no unusual arrangement of subject matter in the text. In the introduction, covering the first twelve pages, the necessary formulas from Algebra and Trigonometry are given for reference. A good number of exercises are given for practice on these important formulas. The definition of and several exercises on the eliminant of a system of linear equations appear in the introduction. In fact, determinants are used freely throughout the text. The definition of the projection of a line is taken up in the introduction and, also, used freely throughout the text.

"The Point" is the subject of Chapter II. The reviewer must include himself among those who will object to the use of two letters, $r_1/r_2$, instead of a single letter, $r$ in expressing the ratio of division in finding the coordinates of a point that divides a line into a given ratio. It is a little unusual to find the definition of the inclination of a line possibly exceeding 180°. Many texts define the angle in a manner not to exceed 180°. The procedure of proving theorems in plane geometry by the analytic method is stressed by the authors. Twenty-one exercises in Chapter II serve to introduce the student to this method. Exercises of this nature are found, also, in Chapters IV and V.

Those very fundamental problems such as "finding the equation of a locus," "finding the locus of an equation," and "discussing the equation of a locus" are taken up in Chapter III. In this Chapter the subjects of asymptotes and intersection points of curves are treated.

In Chapter IV forty-eight pages are devoted to the straight line. The development of the theory is set forth in the form of theorems. It is a pleasure to read this subject matter, both from the standpoint of clarity of language and rigor of analysis.

Chapters V, VI, and VII, are devoted to the conic sections and tangents. In Chapter VIII transformation of coordinates is taken

up. The usual simple ideas of the invariant are left out of this chapter. For Chapter IX the title is "Other Loci". Such topics as Poles, Polars, and Diameters are treated. Polar coordinates are given in Chapter X. An introduction, at least, to Polar Coordinates might better have come earlier in the text. In Chapter XI Parametric Equations are treated.

Chapters XII through XIV are given over to Solid Analytic Geometry. The treatment of the Solid Analytic Geometry is brief. The text closes with Chapter XV which gives an introduction to Calculus with simple applications.

The reviewer is impressed with the apparent teachability of this text. It appears to be adapted especially to Arts students. For the Engineering student who desires to get into his subject more quickly and use his subject as a tool primarily, the text appears not so well adapted. For the Arts student majoring in mathematics the treatise should well be adapted.

The format of the book is especially pleasing. For one interested in selecting a superior text in Analytics the reviewer recommends the treatise for examination.

*Louisiana Polytechnic Institute.* P. K. SMITH.

*College Algebra.* By Hobart C. Carter. *Prentice-Hall,* New York, N. Y., 1936. x+234 pages.

This is not just another freshman algebra text. The preface states that "The definitions, statements, theorems, and proofs are straightforward, general, rigorous, and concise. Every sentence and paragraph is made in view of a definite objective. Thus, the student is given unusually clear and simple explanations and is not confused by unnecessary words and paragraphs." Whether or not we agree with the author that conciseness leads to simplicity we must credit him with painstakingly having eliminated all "unnecessary words and paragraphs." The two most striking features of the book are its conciseness and the large number of topics treated.

The author has not been a slave to precedent in arranging his material. The first chapter contains a development of the number system and a review of elementary algebra leading up to the solution of equations. The notion of *function* and the graphic representation of functions are introduced here and are used consistently thereafter. The next three chapters are concerned with linear, quadratic, and algebraic functions. They include linear and quadratic equations,

second and third order determinants, theorems on the sum and product of the roots of a quadratic, the usual theorems on polynomials in a single variable, isolation of the real roots of an equation, Horner's method, and a section on the graphs of fractional functions. The next chapter is on logarithmic and exponential functions, while the sixth and last chapter is on miscellaneous topics, including variation, permutations and combinations, the binomial theorem, probability, progressions, mathematical induction, determinants, frequency distributions, compound interest, annuities, capitalized cost, annual investment charge, and systems of numeration.

Among the features of the book which the reviewer particularly liked were the numerous historical footnotes (which he only wished were even more copious), and the inclusion of a section on the simpler statistical measurements in connection with frequency distributions.

Some teachers will believe that the aim of conciseness has been carried too far, that in some places clearness would have been gained by the addition of slightly more detailed explanations. The abrupt introduction of new ideas and the omission of details in proofs, however desirable in a book designed for mature readers, is more likely to mystify the first year college student than to save his time. For example, the laws of exponents are assumed for integral exponents (p. 16) and then defined for fractional exponents. The average student would obtain a much better idea of the development of these definitions if the laws were assumed only for positive integral exponents (or even better, if they were defined for positive integral exponents, the laws proved for this case, and the definitions then extended to the other cases). Again, if on page 2 the author had been more explicit about the setting up of the correspondence between real numbers and the points of a line, the definitions in connection with the linear ordering of the real numbers on page 54 would have been clearer. Again, the average student would follow the proof of the binomial theorem by means of combinations (p. 123) with greater ease if treated with more detail.

The book is characterized by the apparent care which has been exercised to make the proofs and statements as rigorous as possible in a work of this kind, a quality often lacking in elementary texts. On page 133 the term infinite is somewhat loosely used in view of the fact that the author's definition of *sum of an infinite progression* precludes an infinite sum. It would have been more exact to say that when $r \geq 1$ or $r \leq -1$, the progression does not possess a sum. On page 138 the author failed to state that steps (1) and (2) constitute a proof for all integers *greater than or equal to that used in step* (1).

At the end of each section there are three sets of exercises of approximately equal difficulty and a fourth set of more theoretical and more difficult exercises for "honor" students. The latter is a particularly admirable feature of the book and teachers will find it most helpful.

The general appearance of the book is good, the type large and clear, and the important formulae are all well displayed. More emphasis would have been added if, in definitions, the words defined had been italicized or put in bold face type.

*College of St. Thomas.* L. E. BUSH.

*Mathematics for Everyday Use.* By John C. Stone and Vergil S. Mallory, State College, Montclair, New Jersey. Ben J. Sanborn & Company, Chicago. xi+532 pages.

This text book is designed principally to give to high school students training in quantitative and functional thinking and the practical knowledge of mathematics needed in everyday life. However, the text provides for those students who might develop an interest in a further study of mathematics by including a supplementary unit in algebra sufficient to prepare students for the second semester of college preparatory algebra.

The main body of the text is divided into the following twelve units: direct measure, graphs, number relations, geometric drawings, volumes of solids, inaccessible measurements, use of equations, positive and negative numbers, social problems, banking service, savings and investments, and insurance.

Some of the desirable features of the text are:

1. The opportunity for pupil activity through measuring, drawing, making models, constructing graphs, etc.
2. The exercises for testing the students ability which encourage him to study the text carefully in order to understand the development from a printed page.
3. The use of graphs in showing quantitative relationships, and the use of graphs in solving problems.
4. Review tests at the end of each unit.
5. The historical notes distributed throughout the text.
6. The printing and the spacing of the material on the pages.
7. Atractive pictures and drawings.

All teachers who believe that all high school students should have at least one unit of mathematics will value this text.

*Louisiana Polytechnic Institute.* HENRY SCHROEDER.